Portfolio analysis using mathematical methods

Agron Gjana[1], MSc.; Glediana Zeneli (Foto)[2], PhD; Arsen Benga[3], MSc.

Abstract

The portfolio with the minimum variance is that portfolio (a proper combination of asset's weights) which, when given the risk and the return of each asset, contains the lowest possible risk level. In other words, the portfolio with a minimum variance specifies the asset weights, by which the lowest risk can be achieved, without any other conditions above the desired return level **(2).** So, it's enough to identify in this case, the lowest existing diversification, since it is directly related to the variance (risk).

The portfolio with a minimum variance is determined by a mathematical optimization based on changing asset weights to reach the lowest possible diversification level. This includes the possibility that some of the weights are zero, so some assets from the portfolio can be eliminated. Portfolio with minimal variance is very important in this portfolio analysis because it has the lowest level of risk reachable from assets that the manager has available. This risk level may result lower than market risk, due to the effects of diversification. Another advantage is the non-inclusion of assets, which do not meet our diversification needs. So if investing in an asset does not reduce the overall portfolio risk, we simply decide to put it out. The minimum variance portfolio serves as an assessor of diversification opportunities.

Key words: Portfolio, minimum variance, Lagrange function

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[1]Department of Mathematics, Faculty of Mathematical and Physical Engineering , Polytechnic University, Albania, email: agron_gjana@yahoo.com

[2] Department of Applied Mathematics, Faculty of Natural Sciences, University of Tirana, Albania, email: gledafoto@yahoo.com

[3] Department of Mathematics, Faculty of Mathematical and Technology , American University of the Middle East , KUWAIT

I. Literature review

In the Markowitz portfolio model, it is assumed that investors choose portfolios based on expected returns $E(r_p)$, and standard deviation of return as a measure of its risk σ_p. Therefore, the problem of portfolio selection can be expressed as maximizing returns based on the given risk (or minimizing risk based on the expected return, maintaining constant returns and choosing weights that minimize variance).

Maximizing returns is probably the main problem for investors. For some of them, this is the only target to be achieved, so they perform poorly in the markets. More cautious investors include risk in one form or another in their investments. The expected risk is as important in an investment as the expected return. As discussed above, portfolio risk is a function of stock returns. There can be no potential for profit without risk potential.

II. Calculating the portfolio with minimum variance

Quantitative maximization of portfolios returns should be carried out including two limitations. The first relates to the establishment of weights of assets equal to 1, while the second relates into fixing the objective risk of the portfolio, being equal to a positive constant. The problem is analogous to the previous risk minimization problem for a certain return level. The optimization problem is again solved by the Lagrange method. In this case, we must determine the weighing vector that maximizes the return to a given level of risk. In solving the problem, we must be careful to meet the two above limitations.

Mathematically, the problem of portfolio choice can be formulated as a quadratic programming. For two risky assets A and B, with portfolio's weights w_A, w_B, the return is measured by:

$$\sigma_P^2 = w_A{}^2\sigma_A{}^2 + w_B{}^2\sigma_B{}^2 + 2w_A w_B \rho_{AB}\sigma_A\sigma_B .$$

Here is the following optimization problem: To find such weights w_A, w_B that minimize the risk, so: $\underset{w_A}{\text{Min}}\ \sigma_P^2 = w_A{}^2\sigma_A{}^2 + w_B{}^2\sigma_B{}^2 + 2w_A w_B \rho_{AB}\sigma_A\sigma_B$.

With the terms: $\begin{cases} w_A + w_B = 1, \\ w_A \geq 0,\ w_B \geq 0 \end{cases}$.

Based on **(2),** the portfolio weights with a minimum variance are given in Table 1

Table1. Portfolio with minimum variance without short sales

Correlation of two assets	A asset weight	B asset weight
ρ= 1	$w_A = \frac{\sigma_B}{\sigma_A - \sigma_B}$	$w_B = \frac{\sigma_A - 2\sigma_B}{\sigma_A - \sigma_B}$
ρ= -1	$w_A = \frac{\sigma_B}{\sigma_A + \sigma_B}$	$w_B = \frac{\sigma_A}{\sigma_A + \sigma_B}$
ρ= 0	$w_A = \frac{\sigma^2_B}{\sigma^2_A + \sigma^2_B}$	$w_B = \frac{\sigma^2_A - 2\sigma^2_B}{\sigma^2_A + \sigma^2_B}$

Above was used the portfolio with two risky assets to calculate the portfolio weights with minimal variance.

If generalized, for portfolios containing N assets, the portfolio weights with minimal variance can be found by minimizing the Lagrange Φ function for portfolio variance.

$$\text{Min } \Phi = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \text{Cov}\left(r_i r_j\right) + \lambda_1\left(1 - \sum_{i=1}^{n} W_i\right)$$

Conditioned by, $\begin{cases} w_1 + w_2 + \ldots + w_n = 1 \\ w_i \geq 0 \quad \forall \text{i=1,2...n} \end{cases}$

Where λ_1 is the Lagrange multiplier, ρ_{ij} is the coefficient of correlation between r_i and r_j, while the other variables are defined as before.

Using this method, the minimum variance can be computed for each expected portfolio return yield. The efficient set generated by the solution of the equation (1) is often called the set of minimum variances, due to the minimizing nature of Lagrange's solution.

One of the portfolio analysis goals is to minimize the risk or portfolio variance

If there are n investments A_1,A_2 ,…A_n with random return level (R_1), (R_2),… R_n with mathematical expected value E(R_1),E(R_2),…E(R_n) and variance, Var(R_1), Var(R_2),… Var(R_n), Var(R_1,R_2),… Var(R_{n-1},R_n), , if we mark by:

E(R_i)=r_i për i=1,2….n and Var(R_1,R_2)= σ_{ij} for i=1,2….n and j=1,2,…n .then the indicators of these investments are the matrices:

$$\begin{bmatrix} r_1 \\ r_2 \\ \cdot \\ \cdot \\ \cdot \\ r_n \end{bmatrix} \text{and} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix}$$

$$E(r_P) = \sum_{i=1}^{n} w_i E(r_i) \qquad \mathrm{Var}(r_p) = \sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \mathrm{Cov}(r_i, r_j)$$

The covariance can also be expressed in terms of the correlation coefficient as follows:

$$\mathrm{Cov}(r_i, r_j) = \rho_{ij}\sigma_i\sigma_j = \sigma_{ij}$$

So: $\mathrm{Var}(r_p) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \rho_{ij}\sigma_i\sigma_j = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}$

Where, $w_i \geq 0$,for i =1,2….n and $\sum_{i=1}^{n} W_i = 1.0$

A portfolio P is characterized by its two indicators: expected return E and variance V. Portfolio P with indicator (E, V) is efficient if:

1. It is therefore permissible $w_i \geq 0$ for I = 1, 2….n .

2 Each portfolio P_1 with variance $V_1<V$ to have an expected return E_1 less than E.

3. Each portfolio P_1 with expected return $E_1> E$ has variance $V_1>V$.

To find the efficient portfolio we act in this way:

At first we find the set of portfolios Γ that meet the condition

$$\min \mathrm{Var}(r_p) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

Where $\begin{cases} \sum_{i=1}^{n} w_i E(r_i) = E^* \\ \sum_{i=1}^{n} w_i = 1 \\ w_i \geq 0 \text{ për i=1,2...n} \end{cases}$

In Γ we find portfolios that meet the condition:

$$\max E(r_P) = \sum_{i=1}^{n} w_i E(r_i)$$

Where $\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} = V^*$

The first condition consists in that the expected return of the portfolio should equal the target return set by the E^* portfolio manager. The second condition relates to the fact that the sum of asset weights should be equal to one.

This is the set of the efficient portfolios Δ. Note that E^* takes random values and V^* takes positive values. The subset Δ of Γ offers the set of efficient portfolios. If the portfolios of set Γ have the quality that for each portfolio P with indicators E, V part of Γ can't be found another efficient portfolio then the set of efficient portfolios is Δ.

Let's solve this problem with the Lagrange multipliers method

Find the lowest value of the function:

$$\mathrm{Var}\left(r_p\right)=\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

Where, $\begin{cases} \sum_{i=1}^{n} w_i E(r_i) - E^* = 0 \\ \sum_{i=1}^{n} w_i - 1 = 0 \end{cases}$

The Lagrange function can be written as:

$$\Phi=\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} + \lambda_1\left[\sum_{i=1}^{n} w_i E\left(r_i\right) - E^*\right] + \lambda_2\left(\sum_{i=1}^{n} w_i - 1\right) \qquad (2)$$

Our problem lies in finding the minimum value of Lagrange's function.

We find the partial derivatives of the equation for each of the variables, $w_1, w_2, \ldots, w_n$, λ_1, λ_2 and equate the zero-generated equations, which provide risk minimization under the Lagrange constraints.

Calculating the partial derivatives we get a system with (n+2) linear equations and (n+2) variables.

$$\begin{cases} \frac{\partial \Phi}{\partial w_1} = 2w_1\sigma_{11} + 2w_2\sigma_{12} + \ldots + 2w_n\sigma_{1n} + \lambda_1 E(r_1) + \lambda_2 = 0 \\ \frac{\partial \Phi}{\partial w_2} = 2w_1\sigma_{21} + 2w_2\sigma_{22} + \ldots + 2w_n\sigma_{2n} + \lambda_1 E(r_2) + \lambda_2 = 0 \\ . \\ . \\ . \\ \frac{\partial \Phi}{\partial w_n} = 2w_1\sigma_{n1} + 2w_2\sigma_{n2} + \ldots + 2w_n\sigma_{nn} + \lambda_1 E(r_n) + \lambda_2 = 0 \\ \frac{\partial \Phi}{\partial \lambda_1} = w_1 E(r_1) + w_2 E(r_2) + \ldots + w_n E(r_n) - E^* = 0 \\ \frac{\partial \Phi}{\partial \lambda_2} = w_1 + w_2 + \ldots + w_n - 1 = 0 \end{cases} \qquad (3)$$

The matrix form of the system with (n+2) linear equations and (n+2) variables looks like.

$$C=\begin{bmatrix} 2\sigma_{11} & 2\sigma_{12} & \dots & 2\sigma_{1n} & E(r_1) & 1 \\ 2\sigma_{21} & 2\sigma_{22} & \dots & 2\sigma_{2n} & E(r_2) & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 2\sigma_{n1} & 2\sigma_{n2} & \dots & 2\sigma_{nn} & E(r_n) & 1 \\ E(r_1) & E(r_2) & \dots & E(r_n) & 0 & 0 \\ 1 & 1 & \dots & 1 & 0 & 0 \end{bmatrix},\ W=\begin{bmatrix} w_1 \\ w_2 \\ . \\ . \\ . \\ w_n \\ \lambda_1 \\ \lambda_2 \end{bmatrix},\ B=\begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ E^* \\ 1 \end{bmatrix}$$

Measurement of the variables $w_{i,}\lambda_{1,}\lambda_2$ s taken from the solution of the matrix equation C*W=B Assuming that the inverse of the matrix C can be found then the vector $W=C^{-1}*B$.

The solution of the system will be a vector with (n+2) components which will be expressed depending on E*.

$$\begin{cases} w_1 = c_1 + d_1 E^* \\ w_2 = c_2 + d_2 E^* \\ . \\ . \\ . \\ w_n = c_n + d_n E^* \end{cases} \qquad (4)$$

Where: $$\sum_{i=1}^{n} w_i = 1$$

For every value of E^*we get a portfolio with a lower variance.

The vector W, defined by the equation C * W = B, is the only one and its constituent components are the minimum points. This is determined by the following theorem.

Theorem:

If Hess's matrix is either positively defined or semi-defined, then W vector satisfying the equation C * W = B is the only one, then its components are the minimum point.

If we look at our matrix C it is positively determined because its main determinants are composed of matrices that determine the variance (we know that random variance is non-negative).

$$C = \left[\begin{array}{cccc|cc} 2\sigma_{11} & 2\sigma_{12} & \dots & 2\sigma_{1n} & E(r_1) & 1 \\ 2\sigma_{21} & 2\sigma_{22} & \dots & 2\sigma_{2n} & E(r_2) & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 2\sigma_{n1} & 2\sigma_{n2} & \dots & 2\sigma_{nn} & E(r_n) & 1 \\ \hline E(r_1) & E(r_2) & \dots & E(r_n) & 0 & 0 \\ 1 & 1 & \dots & 1 & 0 & 0 \end{array}\right]$$

Efficient portfolios can be considered the portfolio that has the maximum return and minimum variance. Efficiency portfolios Γ will be determined by solving this problem:

Find the maximum value of the expected return $E(r_P) = \sum_{i=1}^{n} w_i E(r_i)$ and the minimum value of the risk which is equivalent with the calculation of maximum value for:

$$\text{-Var}(r_p) = -\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

With one condition, $\sum_{i=1}^{n} w_i = 1$

Combining both conditions we take this problem:

Find the greatest value of the function: F(max)= $E(r_p)$-$Var(r_p)$

Lagrange function will be written as:

$$\Phi = \alpha\sum_{i=1}^{n} w_i E(r_i) - \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} + \lambda\left(1 - \sum_{i=1}^{n} w_i\right) \qquad (5)$$

Where α number is a qualitative variable which represents the financial manager's preferences between expected return and risk.

Finding the set of solutions to this problem we get the set of optimal portfolios considering the company's preferences.

To find the maximum point of the Lagrange function we form the system:

$$\begin{cases} \dfrac{\partial \Phi}{\partial w_1} = \alpha E(r_1) - 2w_1\sigma_{11} + 2w_2\sigma_{12} + \ldots + 2w_n\sigma_{1n} - \lambda \ = 0 \\ \dfrac{\partial \Phi}{\partial w_2} = \alpha E(r_2) - 2w_1\sigma_{21} + 2w_2\sigma_{22} + \ldots + 2w_n\sigma_{2n} - \lambda \ = 0 \\ . \\ . \\ . \\ \dfrac{\partial \Phi}{\partial w_n} = \alpha_1 E(r_n) - 2w_1\sigma_{n1} + 2w_2\sigma_{n2} + \ldots + 2w_n\sigma_{nn} - \lambda \ = 0 \\ \dfrac{\partial \Phi}{\partial \lambda} = 1 - w_1 + w_2 + \ldots + w_n \ = 0 \end{cases} \tag{6}$$

The linear system with (n+1) equations and (n+1) variables can be expressed in the matrix form CW=B*

$$C = \begin{bmatrix} 2\sigma_{11} & 2\sigma_{12} & \ldots & 2\sigma_{1n} & E(r_1) & 1 \\ 2\sigma_{21} & 2\sigma_{22} & \ldots & 2\sigma_{2n} & E(r_2) & 1 \\ 1\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 2\sigma_{n1} & 2\sigma_{n2} & \ldots & 2\sigma_{nn} & E(r_n) & 1 \\ 1 & 1 & \ldots & 1 & 0 & 0 \end{bmatrix} W = \begin{bmatrix} w_1 \\ w_2 \\ . \\ . \\ . \\ w_n \\ \lambda \end{bmatrix} B^* = \begin{bmatrix} \alpha E(r_1) \\ \alpha E(r_2) \\ . \\ . \\ . \\ \alpha E(r_n) \\ 1 \end{bmatrix}$$

Assuming that matrix C has an inverse then the solution would be: $W = C^{-1}B^*$

Or: $\begin{cases} w_1 = c_1 + d_1\alpha \\ w_2 = c_2 + d_2\alpha \\ . \\ . \\ . \\ w_n = c_n + d_n\alpha \end{cases}$ where c_i and d_i are constants.

Finally let's give an algorithm for finding efficient portfolio:

We have marked with the w_i weights of each of the investments that make up the portfolio P.

We want to calculate the unknowns w_i that satisfy these conditions:

1. Portfolio P with investments $I_1, I_2, .. I_n$ to have a constant return defined previously.

2. Among these constant return E portfolios, find the one with the lowest risk (the variance has the smallest value).

Let's note:

$$W = \begin{bmatrix} w_1 \\ w_2 \\ . \\ . \\ . \\ w_n \end{bmatrix}, \quad V = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{n1} & \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix}, \quad R = \begin{bmatrix} E(r_1) \\ E(r_2) \\ . \\ . \\ . \\ E(r_n) \end{bmatrix}, \quad e = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ . \\ 1 \end{bmatrix}$$

The above problem can be written as follows:

Find $w_1, w_2, \ldots, w_n$ so:

$$\begin{cases} \sum_{i=1}^{n} w_i E(r_i) = E \\ \sum_{i=1}^{n} w_i = 1 \\ w_i \geq 0 \end{cases}$$

$$\min \mathrm{Var}\left(r_p\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

Or using the matrix form we can rewrite it as follows:

Find the vector W so the quadratic function:

$$\sigma^2(risku) = \mathrm{Var}\left(r_p\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} = W^T V W \quad \text{minimum}$$

Conditioned by: $\begin{cases} R^T W = E \\ e^T W = 1 \\ w_i \geq 0 \end{cases}$

To solve this problem, we use the same idea as above using the method of the Lagrange multipliers.

$$\Phi(w_1, w_2, \ldots, w_n, \lambda_1, \lambda_2) = \Phi(w_i, \lambda_1, \lambda_2) = W^T V W - \lambda_1 (R^T W - E) - \lambda_2 (e^T W - 1)$$

We find the partial derivatives about w_i, λ_1, λ_2 , as a result:

$$\begin{cases} \dfrac{\partial \Phi}{\partial w_i} = 2VW - \lambda_1 R - \lambda_2 e = 0 \\ \dfrac{\partial \Phi}{\partial \lambda_1} = E - R^T W = 0 \\ \dfrac{\partial \Phi}{\partial \lambda_2} = e^T W - 1 = 0 \end{cases}$$

For having the solution:

Mark as K the nx2 matrix:

$$K = \begin{bmatrix} E(r_1) & 1 \\ E(r_2) & 1 \\ . & . \\ . & . \\ . & . \\ E(r_n) & 1 \end{bmatrix}$$

Using matrix blocks, the matrix K is presented as K=$\begin{bmatrix} R, & e \end{bmatrix}$

We mark as P and λ the2x1 and 1x2 matrices respectively

$P = [E, 1]^T \ and \ \lambda = [\lambda_1, \lambda_2]$

By means of these notes the above system is written as:

$$\begin{cases} 2VW - K\lambda = 0 \\ K^T W = P \end{cases}$$

From the solution of the system's first equation we get the vector: $W = \frac{1}{2} V^{-1} K\lambda$.

(Matrix V^{-1}exists since the variance matrix is positively determined)

We replace the vector W found above in the second equation of the system and we will have:

$K^T \left(\frac{1}{2} V^{-1} K\lambda\right) = P \ or \ 2P = k^T V^{-1} K\lambda$

Note by A the 2x2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The solution of the matrix equation shows up that:

A=$K^T V^{-1} K$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} E(r)^T \\ e^T \end{bmatrix} .V^{-1}.\begin{bmatrix} E(r), & e \end{bmatrix}$$

$$a_{11} = E(r)^T V^{-1} E(r)$$
$$a_{12} = a_{21} = E(r)^T .V^{-1} e$$
$$a_{22} = e^T V^{-1} e$$

Since matrix A is positively determined then it has an inverse matrix A^{-1}.

Using the matrix equation $2P = A\lambda$ we find the vector $\lambda=2\ A^{-1}P$. The following equality gives the vector W:

$$W=\frac{1}{|A|}V^{-1}\left[R,\quad e\right]\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}\begin{bmatrix} E \\ 1 \end{bmatrix}$$

$$W=\frac{1}{|A|}V^{-1}\left[E(a_{22}R-a_{12}e)+a_{11}e-a_{12}R\right]$$

Let mark as C and D the vector columns $C=\begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_n \end{bmatrix}$ and $D=\begin{bmatrix} d_1 \\ d_2 \\ . \\ . \\ . \\ d_n \end{bmatrix}$

Based on this we find that the requested portfolio is:

$W = CE + D$ or $wi = ciE + di$ for $i = 1,2,\ldots,n$

In this conditions we have the question: Which is the variance for this portfolio?

$$\sigma^2(risk) = \mathrm{Var}\left(r_p\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} = W^T VW$$

Since $W = V^{-1}KA^{-1}P$, then $\sigma^2 = V = (V^{-1}KA^{-1}P)^T VV^{-1}KA^{-1}P$

We know that $A = K^T V^{-1}K$ or $A^T = K^T(V^{-1})^T K$

Then $\sigma^2 = V = P^T A^{-1}P$

In the same way as we did for vector W we will get:

$$\sigma^2(risku) = \mathrm{Var}\left(r_p\right) = \frac{a_{22}E^2 - 2a_{12}E + a_{11}}{a_{11}a_{22} - (a_{12})^2}$$

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III. Conclusion

We provide the algorithm for determining the efficient area (E, V) as follows:

1. Estimated returns and standard deviations are calculated from the statistical data for investments $E(r_i)$ and σ_{ij} for i =1,2,…,n and j=1,2,…,n

2. Note the ratings received, respectively R and V matrices

3. Calculate the matrix V^{-1}

4. Calculate the vectors $\alpha= V^{-1}R$ and $\beta= V^{-1}E$

5. Calculate the elements of the matrix A:$a_{11}=R^T\alpha$, $a_{12}= a_{21}=R^T\beta$ and $a_{22}= e^T\beta$

6. Calculate the determinant of matrix A, which should always be positive as well as the matrix

V determinant.

7. Calculate the vectors with n-dimensions:

$$C = \frac{1}{|A|}(a_{22}\alpha - a_{12}\beta) \quad \text{dhe} \quad D = \frac{1}{|A|}(a_{11}\beta - a_{12}\alpha)$$

8. Determined the optimal portfolio by the vector W: W=CE-D

The i-th component of this vector determines the investment's weight in i-i's action.

$$Since \sum_{i=1}^{n} w_i = 1.0 \ is \ requested \ that \sum_{i=1}^{n} c_i = 0 \ and \sum_{i=1}^{n} d_i = 1$$

9. Calculate σ^2:

$$\sigma^2(risku) = \mathrm{Var}\left(r_p\right) = \frac{a_{22}E^2 - 2a_{12}E + a_{11}}{|A|}$$

10. Based on the step’s 8 and 9 feedback we make the analysis of the portfolio.

IV. References

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